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ON AN INVERSE PROBLEM FOR NONNEGATIVE AND EVENTUALLY NONNEGATIVE MATRICES[†]

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ABSTRACT

Let $\sigma = \{\lambda_1, \cdots, \lambda_n\} \subset C$. We discuss conditions for which σ is the spectrum of a nonnegative or eventually nonnegative matrix. This brings us to study rational functions with nonnegative Maclaurin coefficients. A conjecture for special sets σ is stated and some evidence in support of this conjecture is given.

1. Introduction

The classical Perron–Frobenius theorem [10, 3] on the spectrum of nonnegative matrices stimulated an enormous number of papers on the one hand, and was applied successfully in various fields of pure and applied mathematics on the other hand. In recent years the following inverse problem became of interest: Give a necessary and sufficient condition for a set σ of n complex numbers $\{\lambda_1, \cdots, \lambda_n\}$ to be a spectrum of a nonnegative $n \times n$ matrix A. See [1], [2], [5], [7–9], [11], [13]. If $A \geqq 0$ then $A^k \geqq 0$ and the obvious necessary conditions are

$$\sum_{j=1}^{n} \lambda_j^k \geqq 0, \tag{1.1}$$

for $k = 1, 2, \cdots$. In the first paper on the subject by Suleimanova [13] it was stated and proved (quite loosely) that if the set σ is real and contains exactly one positive number then the condition $\sum_{j=1}^{n} \lambda_j \geqq 0$ is a necessary and sufficient condition for σ to be the spectrum of a nonnegative matrix. Note that in this case the inequality (1.1) for $k = 1$ implies immediately (1.1) for $k > 1$.

In the general case, however, the conditions (1.1) for $k = 1, 2, \cdots$ are not sufficient for σ to be a spectrum of $A \geqq 0$. This is true even in cases that σ is real

[†] Dedicated to my teacher Professor Menachem Schiffer on the occasion of his sixty-fifth birthday.

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and contains two positive numbers [11]. Indeed, take for example $\sigma = \{1, 1, -\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3}\}$. If σ is the spectrum of $A \geqq 0$, then according to the theorem of Frobenius, A should be reducible. So σ can be split into two disjoint sets $\sigma = \sigma_1 \cup \sigma_2$, where each σ_i satisfies (1.1). This is clearly impossible. The Suleimanova result was reproved and extended by [7–9], [1], [5] and [11] to more general sufficient conditions for a real set σ to be a spectrum of nonnegative (positive) matrices. Recently, Fiedler showed that practically all known sufficient conditions for a real σ are also sufficient for the existence of a nonnegative (positive) symmetric matrix with these eigenvalues.

It is interesting to note that all known sufficient conditions on σ require σ to be real. We conjecture that Suleimanova's result holds without assumptions that σ is real.

CONJECTURE. *Let $\sigma = \{\lambda_1, \cdots, \lambda_n\}$ be a set of n complex numbers. Assume that σ satisfies the conditions* (1.1) *for $k = 1, 2, \cdots$. If σ contains exactly one positive number then σ is the spectrum of some nonnegative $n \times n$ matrix.*

In support of this conjecture we prove:

THEOREM 7. *Let $\sigma = \{\lambda_1, \cdots, \lambda_n\}$ be a set of n complex numbers. Assume that σ satisfies the conditions* (1.1) *for $k \geqq M$. Suppose that σ contains exactly one positive number. Then σ is the spectrum of some real $n \times n$ matrix A, such that $A^k \geqq 0$ for $k \geqq N$.*

Such a matrix A is called eventually nonnegative. In particular Theorem 7 implies the validity of our conjecture if we assume in addition that all $|\lambda_i|$ are equal. We now describe briefly the organization of the paper. In the second section we give a refined version of the classical Pringsheim theorem for rational functions. This theorem is our main tool in investigating sets satisfying the conditions (1.1). In particular we note that the conditions (1.1) for $k \geqq M$ imply that $\max_{1 \leqq j \leqq n} |\lambda_j|$ belongs to σ. In Section 3 we consider a set σ satisfying the conditions (1.1) and which contains exactly one or two distinct positive numbers. In the last section we apply our result to the inverse eigenvalue problem for nonnegative and eventually nonnegative matrices. We also give "natural" sufficient conditions for a set $\sigma \subset \mathbf{C}$ to be a spectrum of a nonnegative matrix. These conditions include the Suleimanova condition.

2. Rational functions with nonnegative Maclaurin coefficients

Let $\sigma = \{\lambda_1, \cdots, \lambda_n\}$ be a set of n (not necessarily distinct) points in the complex plane $\mathbf{C}$. The k-th moment $s_k(\sigma)$ of σ is defined to be

$$s_k(\sigma) = \sum_{j=1}^{n} \lambda_j^k \tag{2.1}$$

for $k = 0, 1, \cdots$. Indeed if μ_σ is a nonnegative measure concentrated at the points $\lambda_1, \cdots, \lambda_n$, i.e.

$$\mu_\sigma = \sum_{j=1}^{n} \delta(z - \lambda_j) \tag{2.2}$$

where $\delta(z - \lambda)$ is the Dirac measure, then

$$s_k(\sigma) = \int_{\mathbf{C}} z^k d\mu_\sigma. \tag{2.3}$$

The sums $s_k(\sigma)$ are generated by the following rational function:

$$f_\sigma(z) = \sum_{j=1}^{n} (1 - \lambda_j z)^{-1} = \sum_{k=0}^{\infty} s_k(\sigma) z^k. \tag{2.4}$$

Let $f(z)$ be an analytic function in the neighbourhood of the origin. Then

$$f(z) = \sum_{k=0}^{\infty} a_k z^k. \tag{2.5}$$

Assume that the radius of convergence $R = R(f)$ of this power series is positive and finite. The classical theorem of Pringsheim states that if $a_k \geqq 0$, for $k = 0, 1, \cdots$, then R is a singular point of f. Since altering a finite number of terms in the series (2.5) does not change the radius of convergence, Pringsheim's theorem holds if we assume that $a_k \geqq 0$ for $k \geqq M$. Consider the function f_σ. It is clear that the radius of convergence of f_σ is $r(\sigma)^{-1}$ where

$$r(\sigma) = \max_{\lambda \in \sigma} |\lambda|. \tag{2.6}$$

We call $r(\sigma)$ the radius of σ. Assume that $s_k(\sigma) \geqq 0$ for $k \geqq M$. From the theorem of Pringsheim we deduce that $z = r(\sigma)^{-1}$ is a singular point of f_σ. Thus we obtain

THEOREM 1. *Let $\sigma = \{\lambda_1, \cdots, \lambda_n\}$ be a set of n complex numbers. If the moments $s_k(\sigma)$ are real and nonnegative for $k \geqq M$ then the radius of σ belongs to the set σ.*

Let f be a rational function. Assume that $0 < R < \infty$. Thus on $|z| = R$, f has poles. Let p be the maximal order of the poles of f on $|z| = R$. We call $z = \lambda$ a p-pole of f if $z = \lambda$ is a pole of order p of f.

DEFINITION 1. The set of triples $\pi(f)=\{(\lambda_1, l_1, p), \cdots, (\lambda_n, l_n, p)\}$ is called the principal part of a rational f if the function $f-\sum_{j=1}^{n} l_j(1-\lambda_j z)^{-p}$ does not have poles of order greater than or equal to $p (p \geqq 1)$ on $|z| = R(f) > 0$. Here

$$(2.7) \qquad l_j \neq 0, \ |\lambda_j|^{-1} = R(f), \ \lambda_j \neq \lambda_k, \quad \text{for} \quad j \neq k, j, k = 1, \cdots, n.$$

Normalize for the sake of convenience:

$$(2.8) \qquad R(f) = 1.$$

In what follows we shall be using frequently the function $e^{2\pi i x}$ mostly where $x = m$ or $x = n$. To simplify the notation, we introduce the convention:

$$(*) \qquad e(x) = e^{2\pi i x}, \qquad \zeta = e^{2\pi i/m}, \qquad \omega = e^{2\pi i/n}.$$

We now give a refined version of Pringsheim's theorem for rational functions which is needed in the sequel.

THEOREM 2. *Let f be a rational function having power series* (2.5) *and normalized by the condition* (2.8). *Let $\{(\lambda_1, l_1, p), \cdots, (\lambda_n, l_n, p)\}$ be the principal part of f. Assume that $a_k \geqq 0$ for $k \geqq M$. Then the point $z = 1$ appears in the principal part of f. Let $\lambda_1 = 1$. Then $l_1 > 0$. Moreover if $n \geqq 2$ then*

$$(2.9) \qquad |l_j| \leqq l_1, \ j = 2, \cdots, n.$$

Assume furthermore that

$$(2.10) \qquad |l_j| = l_1, \ j = 2, \cdots, m, \qquad |l_j| < l_1, \ j = m+1, \cdots, n.$$

Then $\lambda_1, \cdots, \lambda_m$ are the m-th roots of unity. Moreover, after a suitable rearrangement of $\lambda_1, \cdots, \lambda_m$ we have

$$(2.11) \qquad \lambda_j = \zeta^{j-1}, \ l_j = \zeta^{(j-1)q}, \ j = 1, \cdots, m$$

for some integer q. Finally, the principal part $\pi(f)$ is invariant under the rotation by $2\pi/m$. That is if $(\lambda, l, p) \in \pi(f)$ then $(\lambda\zeta, \zeta^q, p) \in \pi(f)$.

PROOF OF THEOREM 2. We may assume that all a_k are nonnegative. Otherwise consider the rational function $f_1 = f - \sum_{k=0}^{M} a_k z^k$. Clearly $\pi(f_1) = \pi(f)$ and f_1 has nonnegative Maclaurin coefficients. By the Pringsheim theorem $z = 1$ is a pole of f. Let p be the order of the pole at $z = 1 = \lambda_1$. It is a standard fact that if $z = \lambda$, $|\lambda| = 1$ is another q-pole of f then $q \leqq p$. This follows from the inequality

$$(2.12) \qquad (1-|z|)^p |f(z)| \leqq (1-|z|)^p f(|z|)$$

for $|z|<1$. As $z=1$ is a p-pole and $a_k \geqq 0$ we deduce

$$l_1 = \lim (1-r)^p f(r), \quad \text{as} \quad r \to 1^-. \tag{2.13}$$

Let λ_j, $|\lambda_j| = 1$ be a p-pole of f. Since

$$|l_j| = \lim (1-r)^p \, |f(r\bar{\lambda})|, \quad \text{as} \quad r \to 1^- \tag{2.14}$$

from (2.12) we deduce the inequality (2.9). Assume now that (2.10) holds and $m \geqq 2$. Put

$$l_j = \eta_j l_1, \ |\eta_j| = 1, \ j = 1, \cdots, m. \tag{2.15}$$

As all a_k are real we have the identity $f(\bar{z}) = \overline{f(z)}$, thus by the Schwarz reflexion principle

$$(\bar{\lambda}_j, \bar{l}_j, p) \in \pi(f), \quad \text{if} \quad (\lambda_j, l_j, p) \in \pi(f). \tag{2.16}$$

Consider the function

$$g(z) = 2f(z) - \bar{\eta}_j f(\bar{\lambda}_j z) - \eta_j f(\lambda_j z) = \sum_{k=0}^{\infty} 2(1 - \operatorname{Re}\{\eta_j \lambda_j^k\}) a_k z^k. \tag{2.17}$$

Clearly $g(z)$ is a rational function, analytic in the unit disc and g may have singularities on $|z|=1$ of order not exceeding p. Moreover g has nonnegative power coefficients. The coefficient of $(1-z)^{-p}$ appearing in $g(z)$ is $2l_1 - \eta_j l_j - \bar{\eta}_j \bar{l}_j = 0$. According to what we showed above g does not have p-poles on the unit circle. This means that the p-pole at $z = \bar{\lambda}_k$, $1 \leqq k \leqq n$, has to disappear in g. Let first $1 \leqq k \leqq m$. Then $(1-\lambda_k z)^{-p}$ appears in f with the coefficient $2l_k$. As $|l_k| = l_1$ the p-pole at $z = \bar{\lambda}_k$ would disappear in $g(z)$ if only

$$\lambda_k = \lambda_j \lambda_{r_1}, \ \lambda_k = \bar{\lambda}_j \lambda_{r_2}, \ \eta_j l_{r_1} + \bar{\eta}_j l_{r_2} = 2l_k, \tag{2.18}$$

where $1 \leqq r_1, r_2 \leqq m$. Let $\lambda_j \neq 1$ be fixed. Then $r_1 \neq r_2$. Thus if k varies from 1 to m r_1 also obtains all the values between 1 and m. This means that the set $\{\lambda_1, \cdots, \lambda_m\}$ is a multiplicative group of order m. So we may assume the normalization $\lambda_j = \zeta^{j-1}$, $j = 1, \cdots, m$, as $\lambda_1, \cdots, \lambda_m$ are distinct.

Consider the last equality in (2.18). As $2l_1 = 2|l_k| \leqq |l_{r_1}| + |l_{r_2}| = 2l_1$ we deduce

$$\eta_j \eta_{r_1} = \eta_k, \qquad \bar{\eta}_j \eta_{r_2} = \eta_k. \tag{2.19}$$

Thus $\{\eta_1, \cdots, \eta_m\}$ is also a multiplicative group. Furthermore, the map $\lambda_j \to \eta_j$ is a homomorphism. So $\eta_j = \eta_2^j$ and $\eta_2 = \zeta^q$ for some $0 \leqq q \leqq m-1$. It is left to prove that if $(\lambda, l, p) \in \pi(f)$ then $(\lambda\zeta, l\zeta^q, p) \in \pi(f)$. Consider the function

$$h(z)=2f(z)-\zeta^q f(\zeta z)-\bar{\zeta}^q f(\bar{\zeta} z). \tag{2.20}$$

As before we deduce that $g(z)$ does not have p-poles on $|z|=1$. So all the poles of $f(z)$ of order p on the unit circle cancel. This is equivalent to the fact that $(\lambda\zeta, l\zeta^q, p)\in\pi(f)$ if $(\lambda, l, p)\in\pi(f)$. The proof of the theorem is completed.

3. Special sets with nonnegative moments

Let $\sigma=\{\lambda_1,\cdots,\lambda_n\}$. The principal part $\pi(\sigma)$ of σ is defined as

$$\pi(\sigma)=\sigma\cap\{z, |z|=r(\sigma)\}. \tag{3.1}$$

The set σ is called simple if $\lambda_j\neq\lambda_k$ for $j\neq k$.

Definition 2. A set $\sigma=\{\lambda_1,\cdots,\lambda_n\}$ is called a Frobenius set if

(i) $r(\sigma)>0$,

(ii) $\pi(\sigma)=\{r(\sigma),\zeta r(\sigma),\cdots,\zeta^{m-1}r(\sigma)\}$ for some $1\leqq m\leqq n$,

(iii) the set σ is invariant under the rotation by an angle $2\pi/m$, i.e. $\zeta\sigma=\sigma$.

The reason we called such a set a Frobenius set is that by the Frobenius theorem the spectral set of a nonnegative irreducible matrix is a Frobenius set. For $r\geqq 0$ denote by σ_r the set

$$\sigma_r=\sigma\cap\{z, |z|=r\}. \tag{3.2}$$

Note that σ_r may be empty.

Theorem 3. *Let $\sigma=\{\lambda_1,\cdots,\lambda_n\}$ be a set of n complex numbers. Assume that eventually all the moments of σ are nonnegative, i.e. $s_k(\sigma)\geqq 0$ for $k\geqq M$. If σ contains exactly one positive number then σ is a Frobenius set.*

Proof. Consider the function f_σ defined by (2.4). From the Pringsheim theorem we deduce that $r(\sigma)\in\sigma$. As σ contains exactly one positive number, $r(\sigma)>0$. Let

$$\begin{aligned} r_1=r(\sigma)=\lambda_1=|\lambda_2|=\cdots=|\lambda_{m_1}|>r_2=|\lambda_{m_1+1}|=\cdots=|\lambda_{m_2}|\\ >\cdots>r_t=|\lambda_{m_{t-1}+1}|=\cdots=|\lambda_{m_t}|, \qquad m_t=n. \end{aligned} \tag{3.3}$$

Thus $\pi(\sigma)=\{\lambda_1,\cdots,\lambda_{m_1}\}$. Note that $\pi(\sigma)$ may contain the same number λ several times.

Let $\pi(f_\sigma)=\{(\mu_1,l_1,1),\cdots,(\mu_m,l_m,1)\}$. Here $\mu_j\neq\mu_k$ for $k\neq j$, l_j is the multiplicity of μ_j in $\pi(\sigma)$. Thus $l_j\geqq 1$. The assumptions of the theorem imply that $\mu_1=r(\sigma)$ and $l_1=1$. By Theorem 2, $l_j=1$, $j=2,\cdots,m$ and $\mu_j=\zeta^{j-1}r(\sigma)$,

$j = 1, \cdots, m$. This means that $m = m_1$ and $\pi(\sigma) = \{r(\sigma), \zeta r(\sigma), \cdots, \zeta^{m-1} r(\sigma)\}$. Assume now that $r_2 > 0$, otherwise the theorem holds trivially. Let $\sigma_2 = \bigcup_{0 \leqq r \leqq r_2} \sigma_r$. Denote

$$f_2 = f_{\sigma_2} = \sum_{k=0}^{\infty} b_k z^k. \tag{3.4}$$

Clearly

$$f_2 = f_\sigma - \sum_{j=1}^{m} (1 - \zeta^{j-1} r(\sigma) z)^{-1}. \tag{3.5}$$

As

$$\sum_{j=1}^{m} \zeta^{(j-1)k} = \begin{cases} 0 & \text{for } k \not\equiv 0 \pmod m, \\ m & \text{for } k \equiv 0 \pmod m). \end{cases} \tag{3.6}$$

We realize that b_k are real and $b_k \geqq 0$ for $k \geqq M$ and $k \not\equiv 0 \pmod m$. As σ contains exactly one positive number, namely $r(\sigma)$, $\pi(\sigma_2) = \{\lambda_{m_1+1}, \cdots, \lambda_{m_2}\}$ does not contain r_2. Let

$$g = f_2 + mA(1 - r_2^m z^m)^{-1} = \sum_{k=0}^{\infty} c_k z^k. \tag{3.7}$$

Clearly $c_k = b_k$ for $k \not\equiv 0 \pmod m$, $c_k = b_k + mAr_2^k$ for $k \equiv 0 \pmod m$. Choose A to be a sufficiently large positive number. Then we have that $c_k \geqq 0$ for $k \geqq M$. We apply now Theorem 2. Clearly $(\zeta^{j-1} r_2, A_j, 1) \in \pi(g)$, where $A_j \geqq A$ for $j = 1, \cdots, m$. As $r_2 \notin \pi(\sigma_2)$ we deduce that $A_1 = A$. According to Theorem 2 $A_j = A$, $j = 2, \cdots, m$. This means that $\zeta^{j-1} r_2 \notin \pi(\sigma_2)$ for $j = 1, \cdots, m$. Moreover $\pi(g)$ is invariant under the rotation by an angle $2\pi/m$. Therefore $\zeta\{\lambda_{m_1+1}, \cdots, \lambda_{m_2}\} = \{\lambda_{m_1+1}, \cdots, \lambda_{m_2}\}$. Thus

$$\sum_{j=m_1+1}^{m_2} \lambda_j^k = 0, \quad \text{for} \quad k \not\equiv 0 \pmod m. \tag{3.8}$$

That is, $b_k = 0$ for $k \not\equiv 0 \pmod m$. Let $\sigma_3 = \bigcup_{0 \leqq r \leqq r_3} \sigma_r$. Considering the function f_{σ_3} we prove in the same manner as for f_{σ_2} that $\{\lambda_{m_2+1}, \cdots, \lambda_{m_3}\}$ is invariant under the rotation by an angle $2\pi/m$. Continuing in the same way we obtain that $\zeta\sigma_r = \sigma_r$ for any $r = r_j$. So $\zeta\sigma = \sigma$ and σ is a Frobenius set. The proof of the theorem is completed.

COROLLARY 1. *Let $\sigma = \{\lambda_1, \cdots, \lambda_n\}$ be a set of n points on the unit circle. Assume that $s_k(\sigma) \geqq 0$, for $k \geqq M$. If the point $z = 1$ appears only once in σ then σ is a set of exactly n roots of unity.*

In case that $\lambda_1, \cdots, \lambda_n$ are algebraic, i.e. each of the λ_j is some root of unity, Corollary 1 was proved independently by M. Newman [6]. Another proof of Corollary 1 was suggested by A. Selberg [12].

It is trivial that Theorem 3 does not hold if we relax the assumption that σ contains exactly one positive number. Indeed let $\sigma = \{1, 1, e^{i\theta}, e^{-i\theta}\}$ where θ is real. Of course, $s_k(\sigma) = 2(1 + \cos k\theta) \geqq 0$, but $e^{i\theta}$ need not be any root of unity. We now examine a set σ satisfying the conditions (1.1) on assumption that σ contains exactly two distinct positive numbers. To do so we need the following theorem.

THEOREM 4. *Let f be a rational function having power series* (2.5) *and normalized by the condition* (2.8). *Let $\{(\lambda_1, l_1, p), \cdots, (\lambda_n, l_n, p)\}$ be the principal part of f. Assume that $\lambda_1 = l_1 = 1$ and all other l_j are positive integers. Suppose that for $k \geqq M$, a_k are real and $a_k \geqq 0$ for $k \not\equiv 0 \pmod m$, when $m > 1$. Assume that $\pi(f)$ is completely uninvariant under the rotation by an angle $2\pi/m$, i.e., if λ is a p-pole on $|z| = 1$ then $\lambda\zeta^q$ is not a p-pole for some $1 \leqq q \leqq m-1$. Then $l_j = 1$ for $2 \leqq j \leqq n$. Let m' be the greatest divisor of m such that all m'-th roots of unity are p-poles of f. Let $m'' = m/m' > 1$. Then there exists r co-prime with m'' such that*

(i) *if m'' is even then the set $\sigma = \{\lambda_1, \cdots, \lambda_n\}$ is equal to the set σ_1 which consists of all $m'r$-roots of unity,*

(ii) *if m'' is odd then either $\sigma = \sigma_1$ or $\sigma = \sigma_1 \cup \sigma_2$ where σ_2 is of the form*

$$\sigma_2 = \bigcup_{q,k,j=1}^{m',r,m''-1} e\left(\left(q + \frac{2k-1}{2r} + \frac{j}{m''}\right)\frac{1}{m'}\right). \tag{3.9}$$

PROOF. We will prove the theorem by induction on m. We divide our proof into 4 steps.

(i) Let $m = 2$. Consider the function

$$f_1(z) = f(z) - f(-z) = \sum_{k=0}^{\infty} 2a_{2k+1}z^{2k+1}. \tag{3.10}$$

Thus $2a_{2k+1} \geqq 0$ for $k \geqq M$. Moreover, from the assumption that if λ is a p-pole $-\lambda$ is not a p-pole of f we deduce that

$$\pi(f_1) = \{(1,1,p),\ (-1,-1,p),\ (\lambda_2, l_2, p),\ (-\lambda_2, -l_2, p), \cdots, (\lambda_n, l_n, p),$$
$$(-\lambda_n, -l_n, p)\}. \tag{3.11}$$

As $l_j \geqq 1$, from Theorem 2 we deduce that $l_j = 1$ for $j = 2, \cdots, n$. Furthermore

$$\pi(f_1) = \bigcup_{j=1}^{2n} (e(j/2n), e(j/2), p),$$

that is, in the notation of Theorem 2, $q = n$. As the coefficient of $(1-\lambda_k z)^{-p}$ in f is positive we deduce that after a suitable rearrangement $\lambda_k = \omega^{k-1}$, $k = 1,\cdots,n$. Since -1 is not a p-pole of f, n is odd. The theorem is established in this case.

(ii) Let $m > 2$ be a prime. We claim that $\bar{\zeta}^q$ is not a p-pole for $1 \leqq q \leqq m-1$. Otherwise consider the function

$$g(z) = 2f(z) - f(\zeta^q z) - f(\bar{\zeta}^q z) = \sum_{k=0}^{\infty} 2(1-\cos 2\pi qk/m)z^k = \sum_{k=0}^{\infty} b_k z^k. \tag{3.12}$$

As a_k are real for $k \geqq M$ we have that ζ^q is also a p-pole of f. So $(\zeta^{\pm q}, l, p) \in \pi(f)$. Obviously $b_k \geqq 0$ for $k \geqq M$. The coefficient of $(1-z)^{-p}$ is $2(1-l) \leqq 0$. According to Theorem 2, $l = 1$ and $g(z)$ does not have p-poles on $|z| = 1$. As $(q, m) = 1$, since m is prime, we deduce that $\pi(f)$ is invariant under the rotation by an angle $2\pi/m$. This contradicts the assumption of the theorem. Let $(\lambda, l, p) \in \pi(f)$ and assume that $\lambda \neq 1$. By the orbit or (λ) we denote all the points of the form $\lambda\zeta^j$, $1 \leqq j \leqq m-1$ such that $\bar{\lambda}\bar{\zeta}^j$ is a p-pole of f. We claim that each $l = 1$ and either or (λ) is empty or contains exactly $m-2$ points. Assume first that for some $1 \leqq j \leqq m-1$, $\lambda\zeta^{\pm j} \notin \text{or}(\lambda)$. So $\bar{\lambda}\zeta^{\pm j} \notin \text{or}(\bar{\lambda})$. Consider the function

$$h(z) = 2f(z) - f(\lambda z) - f(\bar{\lambda} z) = \sum_{k=0}^{\infty} c_k z^k \tag{3.13}$$

where c_k are real and $c_k \geqq 0$ for $k \not\equiv 0 \pmod m$. Note that $\zeta^{\pm j}$ is not a p-pole of $h(z)$. Moreover, the coefficient of $(1-z)^{-p}$ is $2(1-l) \leqq 0$. As in the proof of Theorem 3 consider the function

$$h_1(z) = h(z) + m^p A(1-z^m)^{-p} = \sum_{k=0}^{\infty} d_k z^k \tag{3.14}$$

where A is a positive sufficiently large number. So $d_k \geqq 0$ for $k \geqq M$. Clearly

$$(1, A + 2(1-l), p),\ (\zeta^{\pm j}, A, p) \in \pi(h_1) \tag{3.15}$$

According to Theorem 2 $A + 2(1-l) \geqq A$, so $l = 1$. Furthermore, since $A \gg l_j$ and m is prime, from Theorem 2 we deduce $(\zeta^{\pm k}, A, p) \in \pi(h_1)$, $1 \leqq k \leqq m$. So $\text{or}(\lambda) = \emptyset$. Suppose now that we do not have $1 \leqq j \leqq m-1$ such that $\lambda\zeta^{\pm j} \notin \text{or}(\lambda)$. Then we claim there exists $1 \leqq q \leqq m-1$ such that $\lambda\zeta^q \notin \text{or}(\lambda)$ but $\lambda\zeta^{2q} \in \text{or}(\lambda)$. Indeed, according to the assumptions there exists j, $1 \leqq j \leqq m-1$ such that $\lambda\zeta^j \notin \text{or}(\lambda)$. If $\lambda\zeta^{2j} \in \text{or}(\lambda)$ then $q = j$. Otherwise, let $3 \leqq r \leqq m-1$ be the first integer such that $\mu = \lambda\zeta^{rj} \in \text{or}(\lambda)$. If such r does not exist clearly

$\lambda\zeta^{\pm j} \notin \mathrm{or}(\lambda)$ contrary to what we assumed. If r is even then take $q = jr/2$. If r is odd let $q = j(r+1)/2$. We claim that $\lambda\zeta^{2q} \in \mathrm{or}(\lambda)$. Otherwise $\mu\zeta^{\pm j} \notin \mathrm{or}(\mu)$. According to what we just proved $\mathrm{or}(\mu)$ is empty. This contradicts the fact that $\bar{\lambda}$ is a p-pole of f. Note that since m is odd $\lambda\zeta^{2q} \neq \lambda$. Recall that (λ, l, p), $(\lambda\zeta^{2q}, l', p) \in \pi(f)$ and $\lambda\zeta^{q} \notin \mathrm{or}(\lambda)$, $\bar{\lambda}\bar{\zeta}^{q} \notin \mathrm{or}(\bar{\lambda})$. Consider the function

$$\varphi(z) = 2f(z) + f(\lambda\zeta^{q}z) + f(\bar{\lambda}\bar{\zeta}^{q}z) = \sum_{k=0}^{\infty} \alpha_k z^k. \tag{3.16}$$

Again for $k \geqq M$, α_k are real and $\alpha_k \geqq 0$ for $k \not\equiv 0 \pmod m$. We have

$$(1,2,p),\ (\zeta^{\pm q}, l + l', p) \in \pi(\varphi).$$

Consider the function

$$\varphi_1(z) = \varphi(z) + m^p A(1 - z^m)^{-p} = \sum_{k=0}^{\infty} \beta_k z^k \tag{3.17}$$

where A is a sufficiently large positive number. So $\beta_k \geqq 0$ for $k \geqq M$. Thus $(1, 2 + A, p)$, $(\zeta^{\pm q}, l + l' + A, p) \in \pi(\varphi_1)$. As $l, l' \geqq 1$, according to Theorem 2 $l = l' = 1$. This establishes our assertion that $l_j = 1$ for $2 \leqq j \leqq n$. Moreover, since m is prime we must have $(\zeta^j, 2 + A, p) \in \pi(\varphi_1)$ for $1 \leqq j \leqq m-1$. Thus $\lambda\zeta^j \in \mathrm{or}(\lambda)$ for $1 \leqq j \leqq m-1$ and $j \neq q$, i.e. $\mathrm{or}(\lambda)$ contains $m-2$ points. We examine two cases.

(a) For any p-pole λ of f $\mathrm{or}(\lambda)$ is empty. Let

$$\psi(z) = f(z) - \frac{1}{m}\sum_{j=1}^{m} f(\zeta^j z) = \sum_{k=0}^{\infty} r_k z^k. \tag{3.18}$$

According to the assumptions of the theorem $r_k \geqq 0$ for $k \geqq M$. As $\mathrm{or}(\lambda)$ is empty we have

$$\begin{aligned}\pi(\psi) = \Big\{&\Big(1, \frac{m-1}{m}, p\Big), \Big(\zeta, -\frac{1}{m}, p\Big), \cdots, \Big(\zeta^{m-1}, -\frac{1}{m}, p\Big), \cdots,\\ &\Big(\lambda_n, \frac{m-1}{m}, p\Big), \Big(\lambda_n\zeta, -\frac{1}{m}, p\Big), \cdots, \Big(\lambda_n\zeta^{m-1}, -\frac{1}{m}, p\Big)\Big\}.\end{aligned} \tag{3.19}$$

According to Theorem 2 $\lambda_k = \omega^{k-1}$ for $k = 1, \cdots, n$ after a suitable rearrangement of $\lambda_1, \cdots, \lambda_n$. Clearly $(n, m) = 1$. According to the notation of the theorem $\sigma = \sigma_1$, where $r = n$ and $m' = 1$.

(b) Let $\sigma' = \{\mu_1, \cdots, \mu_r\}$, $r \geqq 1$, such that each μ_j is not a p-pole of f but $\mu_j\zeta^k$ is a p-pole of f for $1 \leqq k \leqq m-1$. Furthermore, let $\sigma'' = \{\lambda_1 = 1, \cdots, \lambda_s\}$ be the rest of the p-poles which are not of the form $\mu_j\zeta^k$. Then if $\lambda \in \sigma''$ the orbit of λ is empty. Let

$$\hat{f}(z) = f(z) - \sum_{j,k=1}^{r,m} (1-\mu_j\zeta^k z)^{-p} = \sum_{k=0}^{\infty} \hat{a}_k z^k. \tag{3.20}$$

Again, for $k \geqq M$ $\hat{a}_k$ are real and $\hat{a}_k \geqq 0$ for $k \not\equiv 0 \pmod m$. We also have

$$\pi(\hat{f}) = \{(1,1,p),\cdots,(\lambda_s,1,p),(\mu_1,-1,p),\cdots,(\mu_r,-1,p)\}. \tag{3.21}$$

Furthermore the orbits of each λ_j and μ_k are empty. Let

$$\hat{\psi}(z) = \hat{f}(z) - \frac{1}{m}\sum_{j=1}^{m} \hat{f}(\zeta^j z) = \sum_{k=0}^{\infty} \hat{r}_k z^k. \tag{3.22}$$

As for ψ $\hat{r}_k \geqq 0$ for $k \geqq M$. Now

$$\begin{aligned}\pi(\hat{\psi}) = \Big\{&\Big(1,\frac{m-1}{m},p\Big),\Big(\zeta,-\frac{1}{m},p\Big),\cdots,\Big(\zeta^{m-1},-\frac{1}{m},p\Big),\cdots,\Big(\lambda_s,\frac{m-1}{m},p\Big),\\ &\Big(\lambda_s\zeta,-\frac{1}{m},p\Big),\cdots,\Big(\lambda_s\zeta^{m-1},-\frac{1}{m},p\Big),\Big(\mu_1,-\frac{m-1}{m},p\Big),\\ &\Big(\mu_1\zeta,\frac{1}{m},p\Big),\cdots,\Big(\mu_1\zeta^{m-1},\frac{1}{m},p\Big),\cdots,\Big(\mu_r,-\frac{m-1}{m},p\Big),\cdots,\Big(\mu_r\zeta^{m-1},\frac{1}{m},p\Big)\Big\}.\end{aligned} \tag{3.23}$$

According to Theorem 2 $s = r$ and $\sigma'' = \{1,\xi^2,\cdots,\xi^{2(r-1)}\}$, $\xi = e(1/2r)$, $\sigma' = \{\xi,\cdots,\xi^{2r-1}\}$. The theorem is verified in this case ($m' = 1$).

(iii) Let m be not a prime and suppose that $m' = 1$. We claim that the orbit of $\lambda = 1$ is empty. Assume to the contrary that $(\zeta^q, l, p) \in \pi(f)$. Consider the function $g(z)$ defined by (3.12). As before we conclude that $l = 1$ and $\pi(f)$ is invariant under the rotation by an angle $2\pi q/m$. Let $(q,m) = q'$, $1 \leqq q' < m$. Then all m/q' roots of unity are p-poles of f, contrary to the assumption $m' = 1$. Let $m = m_1m_2$ where $1 < m_1$, m_2. Clearly $a_k \geqq 0$ for $k \not\equiv 0 \pmod{m_2}$. Let us decompose $\pi(f)$ to $\pi_1 \cup \pi_2$ where π_2 is invariant under the rotation by $2\pi/m_2$ and π_1 is completely uninvariant under the rotation. Obviously $\pi_1 \neq \emptyset$.

(a) Assume first that m is even. Then we choose m_2 to be even. By the induction hypothesis

$$\pi_1 = \{(1,1,p),(\eta,1,p),\cdots,(\eta^{r-1},1,p)\},\ \eta = e(1/r). \tag{3.24}$$

As $m' = 1$ we deduce $(r,m) = 1$. Suppose that $\pi_2 \neq \emptyset$. Thus

$$\pi_2 = \{(\nu_1, t_1, p), (\nu_1\zeta_2, t_1, p), \cdots, (\nu_1\zeta_2^{m_2-1}, t_1, p), \cdots,$$

$$(\nu_s, t_s, p), \cdots, (\nu_s\zeta_2^{m_2-1}, t_s, p)\}, \qquad \zeta_2 = e(1/m_2), \tag{3.25}$$

where $t_j \geqq 1$, $1 \leqq j \leqq s$. Let

$$\theta(z) = m_2^{-p} \sum_{j=1}^{m_2} f(\zeta_2^j z^{1/m_2}) = m_2^{-p+1} \sum_{k=0}^{\infty} a_{m_2k} z^k \tag{3.26}$$

where a_{m_2k} are real and $a_{m_2k} \geqq 0$ for $k \not\equiv 0 \pmod{m_1}$ if $k \geqq M$. Let

$$\pi(\theta) = \{(1, 1, p), (\mu_2, \rho_2, p), \cdots, (\mu_u, \rho_u, p)\}. \tag{3.27}$$

If $\mu = \eta^j, 1 \leqq j \leqq r-1$ then either $\rho = 1$ or $\rho = m_2t_k + 1$ in case that $\nu_k^{m_2} = \eta^j$. If μ is not r-th root of unity then $\mu = \nu_k^{m_2}$ and $\rho = m_2t_k$. Let

$$\theta(z) = \theta_1(z) + \theta_2(z) \tag{3.28}$$

where $\theta_1(z)$ satisfies the assumptions of the theorem and $\pi(\theta_2)$ is invariant under the rotation of $2\pi/m_1$. Suppose that $\mu = \eta^j$, $1 \leqq j \leqq r-1$ and $\rho = m_2t_k + 1$. As $t_k \geqq 1$ and $m_2 \geqq 2$ from the induction hypothesis for $\pi(\theta_1)$ we deduce that $(\mu, \rho', p) \in \pi(\theta_2)$, where either $\rho' = m_2t_k$ or $\rho' = m_2t_k + 1$. The invariance of $\pi(\theta_2)$ implies that $(\nu_k^{m_2}\zeta_1^j, \rho', p) \in \pi_2$, $1 \leqq j \leqq m_1$, $\zeta_1 = e(1/m_1)$. From the induction hypothesis for $\pi(\theta_1)$ we deduce that $\rho' \geqq \rho - 1 = m_2t_k > 1$. This means that $\nu_k^{m_2}\zeta_1^j = \nu_{k(j)}^{m_2}$ for any $1 \leqq j \leqq m_1 - 1$. So $\bar{\nu}_k\bar{\zeta}^j$ are p-poles of f for $1 \leqq j \leqq m$, contrary to the uninvariance of $\pi(f)$. Suppose that $\nu_k^{m_2}$ is not r-th root of unity. If $\rho' > 1$ then we will have a contradiction as before. Assume that all ρ' are equal to 1. Thus $m_2 = 2$, $t_k = 1$ and $\rho' = \rho - 1$. If $\nu_k^2\zeta_1^j = \nu_{k_j}^2$ for any $1 \leqq j \leqq m_1 - 1$ we will have a contradiction as before. So for some j, $1 \leqq j \leqq m-1$, $\nu_k^2\zeta_1^j = \eta^{2v}$. This happens only for one j, otherwise we would get that or $(1) \neq \emptyset$. Therefore we conclude that the orbit of ν_k contains exactly $m-2$ points. Furthermore, if $\eta^j \notin \text{or}(\nu_k)$, $1 \leqq k \leqq s$, then the orbit of η^j may contain roots of unity. Since $\text{or}(1) = \emptyset$ we deduce that $\text{or}(\eta^j) = \emptyset$. Thus we arrived at the situation described in (ii b). As in (ii b) we deduce that $\sigma = \sigma_1 \cup \sigma_2$ with $m' = 1$. Since m is even and r is odd ($(m, r) = 1$) it is easy to show that $l_1 > 1$ contrary to our assumptions. So $\pi_2 = \emptyset$ and $\sigma = \sigma_1$ in the notation of the theorem.

(b) Let m be odd. So $m_2 \geqq 3$. If π_1 is of the form (3.24) as in (iii a) we deduce that $\pi_2 = \emptyset$ and the theorem is proved. Assume that $\pi_1 = \pi_3 \cup \pi_4$ where π_3 is of the form (3.24) and

$$\pi_4 = \bigcup_{k,j=1}^{r, m_2-1} \left(e\left(\frac{2k-1}{2r} + \frac{j}{m_2}\right), 1, p \right). \tag{3.29}$$

As $m'=1$, $(r,m)=1$. If $\pi_2=\emptyset$ we finished the proof. Let $\pi_2\neq\emptyset$. Then π_2 is of the form (3.25). Consider $\theta(z)$ defined by (3.26). Thus $\pi(\theta)$ is given by (3.27). If $\mu=\eta^j$ then either $\rho=1$ or $\rho=m_2t_k+1$ in case that $\nu_k^{m_2}=\eta^j$. If $\mu=\xi^{2j-1}$ ($\xi=e(1/2r)$) then either $\rho=m_2-1$ or $\rho=m_2(t_k+1)-1$ in case that $\nu_k^{m_2}=\zeta^{2j-1}$. For other μ which equal $\nu_k^{m_2}$, $\rho=m_2t_k$. As in (iii a) decompose θ to $\theta_1+\theta_2$. If $t_k>1$ then $\rho'\geqq\rho-1\geqq 2m_2-1>m_2-1$, where $(\nu_k^{m_2},\rho',p)\in\pi(\theta_2)$. As $(\nu_k^{m_2}\zeta_1^j,\rho',p)\in\pi(\theta_2)$ for $1\leqq j\leqq m_1$ we conclude that $\bar{\nu}_k\bar{\zeta}^j$ is a p-pole of f for $1\leqq j\leqq m$. This contradicts the uninvariance of $\pi(f)$. Thus each $t_k=1$. Assume first that $\rho'=m_2+1$ for some ν_k. In particular $\nu_k^{m_2}=\eta^v$. Since $\pi(\theta_2)$ is invariant under the rotation by $2\pi/m_1$ and $\pi(f)$ is uninvariant we must have $\nu_k^{m_2}\zeta_1^j=\xi^{2w-1}$ for some $1\leqq j\leqq m_1-1$. Thus $\zeta_1^j=\xi^{2w-2v-1}$. This is impossible as m_1 is odd. In the same way we eliminate a possibility that either $\nu_k^{m_2}=\xi^{2j-1}$ or $\rho'=m_2$. So we are left with the possibility that $\rho'=m_2-1$. Moreover each orbit of $\nu_k^{m_2}$ with respect to m_1 contains exactly one point of the form ξ^{2j-1} as $(r,m)=1$. Suppose that there exists a point ξ^{2j-1} which does not belong to any m_1 orbit of $\nu_k^{m_2}$. So $(\xi^{2j-1},\rho'',p)\in\pi(\theta_2)$ where either $\rho''=m_2-1$ or $\rho''=m_2-2$. Thus the m_1 orbit of ξ^{2j-1} contains either ξ^{2k-1} or η^w. This is impossible since $(m,r)=1$ and m_1 is odd. Thus we proved

$$\pi(\theta_1)=\{(1,1,p),\cdots,(\eta^{r-1},1,p),\ (\nu_1^{m_2},1,p),\cdots,(\nu_s^{m_2},1,p)\},$$

$$\pi(\theta_2)=\{(\xi,m_2-1,p),\cdots,(\xi^{2r-1},m_2-1,p),\ (\nu_1^{m_2},m_2-1,p),\cdots,\\ (\nu_s^{m_2},m_2-1,p)\}, \tag{3.30}$$

$$\pi(\theta_2)=\bigcup_{j,k=1}^{r,m_1}\left(e\left(\frac{2j-1}{2r}+\frac{k}{m_1}\right),m_2-1,p\right). \tag{3.31}$$

This establishes the equality $\sigma=\sigma_1\cup\sigma_2(m'=1)$.

(iv) Let m be not a prime and suppose that $(\xi^q,l,p)\in\pi(f)$. By considering the function $g(z)$ defined in (3.12) we deduce that $l=1$ and $\pi(f)$ is invariant under the rotation by $2\pi/m_1$ where $m_1=m/(q,m)$. Thus all m-th roots of unity which are p-poles of f constitute a subgroup of order m' and $\pi(f)$ is invariant under the rotation by $2\pi/m'$. Let $m'>1$. Consider the function $\theta(z)$ given by (3.26) where $m_2=m'$. The function θ satisfies the assumptions of step (iii) of our proof. Using the results of (iii) we easily deduce the theorem. The proof of the theorem is completed.

From Theorem 3 and 4 we deduce:

THEOREM 5. *Let $\sigma' = \{\lambda_1, \cdots, \lambda_{n'}\}$ be a set of n' complex numbers. Assume that σ' contains exactly two distinct positive numbers $r(\sigma') > r_0 > 0$. Assume for convenience that $r_0 = 1$. Suppose that $s_k(\sigma) \geqq 0$ for $k \geqq M$. Then*

$$\pi(\sigma') = \bigcup_{1 \leqq j \leqq m} r(\sigma')\zeta^j.$$

Assume that $m > 1$. Let $1 < r < r(\sigma')$. If σ'_r is not empty then σ'_r is invariant under the rotation by $2\pi/m$. Let σ'' be the maximal subset of $\sigma_0 = \sigma' \cap \{z, |z| = 1\}$ which is invariant under the rotation by $2\pi/m$. Let $\sigma_0 = \sigma \cup \sigma''$ and assume that $\sigma = \{\lambda_1, \cdots, \lambda_n\}$ is not empty, i.e. $1 \in \sigma$. Then σ is of the form described in Theorem 4. That is, let m' be the greatest divisor of m such that σ is invariant under the rotation by $2\pi/m'$. Then there exists r, co-prime with $m'' = m/m'$, such that

(i) *if m'' is even then $\sigma = \sigma_1$, where $\sigma_1 = \bigcup_{1 \leqq j \leqq m'r} e(j/m'r)$,*

(ii) *if m'' is odd then either $\sigma = \sigma_1$, or $\sigma = \sigma_1 \cup \sigma_2$ where σ_2 is of the form* (3.9).

We conclude this section with an open problem.

PROBLEM. Let $\sigma = \{\lambda_1, \cdots, \lambda_n\}$ be a simple set in **C**, i.e. $\lambda_j \neq \lambda_k$ for $j \neq k$. Assume that $s_k(\sigma) \geqq 0$ for $k \geqq M$. Find the structure of σ.

Theorems 3 and 5 answer our problem in case the set σ is concentrated on one or two circles.

4. Nonnegative and eventually nonnegative matrices

Let A be an $n \times n$ real valued matrix. We call A eventually nonnegative if $A^k \geqq 0$ for $k \geqq M$. Let $\sigma = \{\lambda_1, \cdots, \lambda_n\}$. Denote by $\tau_k(\sigma)$ the k-th symmetric polynomial in $\lambda_1, \cdots, \lambda_n$

$$\tau_k(\sigma) = \sum_{1 \leqq j_1 < \cdots < j_k \leqq n} \lambda_{j_1} \cdots \lambda_{j_k}, \qquad k = 1, \cdots, n. \tag{4.1}$$

We call σ a self-conjugate set if $\tau_k(\sigma)$ are real for $1 \leqq k \leqq n$.

From the Perron–Frobenius theorem we easily obtain:

LEMMA 1. *Let A be an eventually nonnegative matrix. If A is not nilpotent then the spectrum of A is a union of self-conjugate Frobenius sets.*

The converse of this lemma is also true.

THEOREM 6. *Let $\sigma = \{\lambda_1, \cdots, \lambda_n\}$ be a union of self-conjugate Frobenius sets. Then there exists an $n \times n$ eventually nonnegative matrix A such that $\sigma = \sigma(A)$.*

PROOF. Clearly it is enough to consider the case where σ itself is a self-conjugate Frobenius set. Without any restriction we may assume that $r(\sigma)=1$.

(i) Consider first the case where $\sigma=\{1,\lambda_2,\cdots,\lambda_n\}$ and $|\lambda_j|<1$ for $2\leqq j\leqq n$. Let $A(\sigma)$ be the companion matrix corresponding to σ, i.e.,

$$A(\sigma)=(a_{ij}(\sigma))_1^n,\ a_{ij}(\sigma)=\delta_{i+1,j}\quad\text{for}\quad 1\leqq i\leqq n-1,\ 1\leqq j\leqq n, \tag{4.2}$$
$$a_{nj}(\sigma)=(-1)^{n-j}\tau_{n-j+1}(\sigma),\ 1\leqq j\leqq n.$$

The assumption that $1\in\sigma$ implies

$$\sum_{j=1}^{n} a_{nj}(\sigma)=1. \tag{4.3}$$

As σ is self conjugate $A(\sigma)$ is real. We have $A(\sigma)u=u$ and $A'(\sigma)v=v$ where $u=(1,1,\cdots,1)$, $v=(v_1,\cdots,v_n)$ and A' denotes the transposed matrix of A. As $\lambda=1$ is a simple eigenvalue of $A(\sigma)$ we may normalize v such that $\Sigma_{i=1}^n v_i=n$. Let J be an $n\times n$ matrix having every element $1/n$.

Consider $X=(1-\rho)J+\rho I$ where I stands for the identity matrix. So $Xu=u$, $Xv=(1-\rho)u+\rho v=w=(w_1,\cdots,w_n)$. Choose $\rho>0$ such that w is positive and X is nonsingular. Let $B=X^{-1}AX$. So $Bu=u$ and $B'w=w$. Clearly $\sigma(B)=\sigma$. As $|\lambda_j|<1$ for $2\leqq j\leqq n$ we have $B^k\to C=(c_{ij})_1^n$ as $k\to\infty$, where $c_{ij}=w_j/\Sigma_{k=1}^n w_k>0$. Thus B is eventually nonnegative and the theorem is proved.

(ii) Let $\pi(\sigma)=\{1,\zeta,\cdots,\zeta^{m-1}\}$ where $m>1$. We may also assume that $0\notin\sigma$, since the zero eigenvalue corresponds to zero matrix. So $n=mn'$ and $\tau_k(\sigma)=0$, if $k\not\equiv 0\ (\mathrm{mod}\ m)$. Let $\sigma'=\{\mu_1,\cdots,\mu_{n'}\}$ be the unique set such that

$$\tau_k(\sigma')=\tau_{mk}(\sigma),\qquad k=1,\cdots,n'. \tag{4.4}$$

Clearly σ' is self conjugate and after a suitable rearrangement we have that $\mu_1=1$ and $|\mu_j|<1$ for $1\leqq j\leqq n'$. According to (i) there exists $n'\times n'$ eventually nonnegative matrix B such that $\sigma'=\sigma(B)$. Let $A=(A_{ij})_1^m$ be the $n\times n$ matrix composed of m^2 block matrices A_{ij} of size $n'\times n'$. Here

$$A_{ij}=\delta_{i+1,j}I,\ 1\leqq i\leqq n-1,\ 1\leqq j\leqq n,\ A_{nj}=\delta_{1j}B,\ 1\leqq j\leqq n. \tag{4.5}$$

In view of (4.4) it is easy to show that $\sigma(A)=\sigma$. It is enough to note that A^m is a block diagonal matrix diag $\{B,\cdots,B\}$. Furthermore $A^k=(A_{ij}^{(k)})_1^n$ where $A_{ij}^{(k)}$ is either zero or $B^{r(i,j)}$ where $r(i,j)\geqq k/m$. Thus, as B is eventually nonnegative, A is also eventually nonnegative. The proof of the theorem is completed.

Combining Theorem 3 with Theorem 6 we obtain

THEOREM 7. *Let $\sigma = \{\lambda_1, \cdots, \lambda_n\}$ be a set of n complex numbers. Assume that $s_k(\sigma) \geqq 0$ for $k \geqq M$. Suppose that σ contains exactly one positive number. Then σ is a spectrum of some real $n \times n$ matrix A, such that $A^k \geqq 0$ for $k \geqq N$.*

In view of Theorem 4 and Lemma 1, Theorem 7 is false if we shall assume that σ contains exactly two distinct positive numbers. From Corollary 1 we deduce

COROLLARY 2. *Let σ satisfy the assumptions of Theorem 7. If $|\lambda_j| = r(\sigma)$ for $1 \leqq j \leqq n$ then σ is the spectrum of the matrix $A = r(\sigma)P$ where, for example, $P = (p_{ij})_1^n$ is a permutation matrix $p_{ij} = \delta_{i+1,j}$, $1 \leqq i, j \leqq n$ $(n+1 \equiv 1)$.*

We give now a simple condition for a self-conjugate set σ to be a spectrum of a nonnegative matrix, which quite surprisingly was overlooked by other authors.

THEOREM 8. *Let $\sigma = \{\lambda_1, \cdots, \lambda_n\}$ be a self-conjugate set. Assume that*

$$(-1)^{k-1}\tau_k(\sigma) \geqq 0, \qquad k = 1, \cdots, n. \tag{4.6}$$

Then σ is the spectrum of the companion matrix $A(\sigma)$, (4.2), which is nonnegative.

We claim that Suleimanova's condition implies (4.6).

LEMMA 2. *Let $\{\mu_1, \cdots, \mu_{n-1}\}$ be nonnegative numbers. Assume that*

$$\sum_{j=1}^{n-1} \mu_j \leqq 1. \tag{4.7}$$

Then $\sigma = \{1, -\mu_1, \cdots, -\mu_{n-1}\}$ satisfies the conditions (4.6). Moreover, the companion matrix $A(\sigma)$ is stochastic.

PROOF. Let $\sigma' = \{\mu_1, \cdots, \mu_{n-1}\}$. Then

$$\begin{aligned} &\tau_k(\sigma) = (-1)^{k-1}[\tau_{k-1}(\sigma') - \tau_k(\sigma')], \qquad 2 \leqq k \leqq n-1, \\ &\tau_1(\sigma) = 1 - \sum_{j=1}^{n-1} \mu_j, \ \ \tau_n(\sigma) = (-1)^{n-1}\tau_{n-1}(\sigma'). \end{aligned} \tag{4.8}$$

Taking into account (4.7) it is enough to show

$$\tau_k(\sigma') \geqq \tau_{k+1}(\sigma') \qquad \text{for} \quad 1 \leqq k \leqq n-2. \tag{4.9}$$

The last inequalities follow directly from the classical Maclaurin inequalities [4, p. 52]

$$1/(n-1) \geqq \tau_1/(n-1) \geqq \left[\tau_2 \Big/ \binom{n-1}{2}\right]^{1/2} \geqq \cdots \geqq (\tau_{n-1})^{1/(n-1)}, \tag{4.10}$$

where $\tau_k = \tau_k(\sigma')$. Indeed,

$$\tau_k - \tau_{k+1} \geqq \tau_k - \left[\tau_k \Big/ \binom{n-1}{k}\right]^{k+1/k} \binom{n-1}{k+1}$$

$$= \tau_k \Big/ \binom{n-1}{k} \left\{ \binom{n-1}{k} - \left[\tau_k \Big/ \binom{n-1}{k}\right]^{1/k} \binom{n-1}{k+1} \right\}$$

$$\geqq \tau_k \Big/ \binom{n-1}{k} \left\{ \binom{n-1}{k} - \frac{1}{n-1} \binom{n-1}{k+1} \right\} \geqq 0.$$

So $(-1)^{k-1}\tau_k(\sigma) \geqq 0$. As $1 \in \sigma$, $\Sigma_{j=1}^n a_{nj}(\sigma) = 1$ and thus $A(\sigma)$ is stochastic. End of Proof.

We conclude our paper with the following observation. Let σ be a simple set such that $s_k(\sigma) \geqq 0$ for $k \geqq M$. According to Theorem 4 and Lemma 1 σ may not be a spectrum of any eventually nonnegative matrix. However, the following result holds:

THEOREM 9. *Let $\sigma = \{\lambda_1, \cdots, \lambda_n\}$ be a simple set, i.e. $\lambda_j \neq \lambda_k$ for $j \neq k$. Assume that $s_k(\sigma) \geqq 0$ for $k \geqq M$. Then there exists a cone K with interior such that $A(\sigma)K \subset K$ (note that $A(\sigma)$ is real). Furthermore $K \subset R^n_+$.*

PROOF. Without any restriction we may assume that $0 \notin \sigma$ (otherwise consider σ after reducing the zero eigenvalue). Let

$$u^k = (s_k(\sigma), s_{k+1}(\sigma), \cdots, s_{k+n-1}(\sigma)), \qquad k \geqq 0. \tag{4.11}$$

The classical identities

$$s_{k+n}(\sigma) = \sum_{j=1}^{n} (-1)^{j-1} \tau_j(\sigma)\, s_{k+n-j}(\sigma), \qquad k \geqq 0 \tag{4.12}$$

imply that

$$A(\sigma)u^k = u^{k+1}, \qquad k \geqq 0. \tag{4.13}$$

From the assumptions of the theorem we have that $u^k \geqq 0$ for $k \geqq M$. Let K be a closure of finite nonnegative combinations of the vectors $\{u^k\}_M^\infty$. As $K \subset R^n_+$, K is a cone. It is left to show that K contains n linearly independent vectors. Consider a matrix $B = (b_{ij})_1^n$ with the columns $u^k, u^{k+1}, \cdots, u^{k+n-1}$. So $b_{ij} = s_{j+i+k-2}(\sigma)$. Let $W_k = (\lambda_i^{k+j-1})_1^n$. A straightforward calculation shows that $B = W_0' W_k$. Thus $|B| = |W_0|^2 \Pi_{j=1}^n \lambda_j^k$, where $|B|$ stands for the determinant of B. Since $\lambda_p \neq \lambda_q$ for $p \neq q$ and $\lambda_p \neq 0$ we deduce that $|B| \neq 0$. Thus $u^k, u^{k+1}, \cdots, u^{k+n-1}$ are linearly independent and K has interior. The proof of the theorem is completed.

It is interesting to note that the conditions (4.6) are equivalent to the fact that u^{k+n} belongs to the cone generated by the vectors $u^k, u^{k+1}, \cdots, u^{k+n-1}$, in the case that these vectors are linearly independent.

REFERENCES

1. P. G. Ciarlet, *Some results in the theory of nonnegative matrices*, Linear Algebra and Appl. **1** (1968), 139–152.

2. M. Fiedler, *Eigenvalues of nonnegative symmetric matrices*, Linear Algebra and Appl. **9** (1974), 119–142.

3. G. Frobenius, *Über Matrizen aus nicht negativen Elementen*, Sitzungsber. Preuss. Akad. Wiss. Berlin Math. Naturwiss, Kl. (1912), 456–477.

4. G. H. Hardy, J. E. Littlewood and G. Po'lya, *Inequalities*, second edition, Cambridge University Press, 1952.

5. R. B. Kellog, Martices similar to a positive or essentially positive matrix, Linear Algebra and Appl. **4** (1971), 191–204.

6. M. Newman, *Nonnegative sums of roots of unity*, National Bureau of Standards, preprint.

7. H. Perfect, *On positive stochastic matrices with real characteristic roots*, Proc. Camb. Phil. Soc. **48** (1952), 271–276.

8. H. Perfect, *Methods of constructing certain stochastic matrices*, Duke Math. J. **20** (1953), 395–404.

9. H. Perfect, *Methods of constructing certain stochastic matrices II*, Duke Math. J. **22** (1955), 305–311.

10. O. Perron, *Über Matrizen*, Math. Ann. **64** (1907), 248–263.

11. F. L. Salzmann, *A note on eigenvalues of nonnegative matrices*, Linear Algebra and Appl. **5** (1972), 329–338.

12. A. Selberg, private communication.

13. K. R. Suleimanova, *Stochastic matrices with real characteristic values*, Dokl. Akad. Nauk SSSR **66** (1949), 343–345.

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